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The nonlinear capillary instability of a liquid jet. Part 1. Theory

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Nonlinear capillary instability of an axisymmetric infinite liquid column is investigated with an initial velocity disturbance consisting of a fundamental and one harmonic component. A third-order solution is developed using the method of strained co-ordinates. For the fundamental disturbance alone, the solution shows that a cut-off zone of wavenumbers (k) exists such that the surface waves grow exponentially below the cut-off zone, linearly in the middle of the zone (near $k = 1$), and an oscillatory solution exists for wavenumbers above the boundary of the zone. For an input including both the fundamental and a harmonic, all wave components grow exponentially when the fundamental is below the cut-off zone. Using a Galilean transformation, the solution is applied to a progressive jet issuing from a nozzle. The jet breaks into drops interspersed with smaller (satellite) drops for $k < 0{\cdot}65$; no satellites exist for $k > 0{\cdot}65$. It is shown theoretically that the formation of satellites can be controlled by forcing the jet with a suitable harmonic added to the fundamental.

1. Introduction

The linearized theory for the breakup of an inviscid liquid column with surface tension was provided by Rayleigh (1945). He showed that (i) the capillary jet is stable for all purely non-axisymmetric disturbances and (ii) the jet is stable or unstable to axisymmetric disturbances, depending on whether the wavelength is less or greater than the circumference of the undisturbed cylinder. He also showed that the unstable waves grow exponentially (e^{qt}) and gave the dispersion relation

$$q^2 = \frac{\sigma}{\rho r_0^3}\frac{I_1(k)}{I_0(k)}k(1-k^2), \tag{1}$$

where σ is the coefficient of surface tension, ρ the density, r_0 the jet radius, I_n is the modified Bessel function of the first kind and k is the wavenumber. It is quite remarkable that the linear theory predicts the jet breakup time reasonably well for monochromatic initial disturbances, even though it fails to describe the jet profile near breakup. In a number of applications, however, the actual profile at breakup and the relation of this profile to the nature of an applied disturbance are of crucial importance.

Donnelly & Glaberson (1966) showed experimentally that the main drops are interspersed with smaller drops (satellites), which could not be accounted for by linear theory. Figure (plate 1) shows that the satellites can form in several different ways depending on the nature of the applied initial disturbance.

Using the method of strained co-ordinates, Yuen (1968) developed a solution for an inviscid jet in the form of an infinite series and carried out the solution up to third order for a monochromatic initial displacement of the free surface. Though not mentioned specifically, Yuen's theory predicts the existence of satellites for all wavenumbers which Goedde & Yuen (1970) discussed in their experimental results. Subsequently, Nayfeh (1970) applied the method of multiple scales to the same problem that Yuen treated and obtained two second-order expansions, one valid for wavenumbers below and above the cut-off and the other for wavenumbers near the cut-off. Nayfeh and Yuen both predicted that the cut-off wavenumber is amplitude dependent. According to the linear theory, the cut-off wavenumber is always one. Yuen gave the cut-off wavenumber $k_c = 1+9\epsilon^2/16$, while Nayfeh gave the cut-off wavenumber $k_c = 1+3\epsilon^2/4$, where ϵ is the non-dimensional wave amplitude. Lafrance (1975), following an analysis similar to Yuen's, developed a third-order solution. Up to the second order, his solution is the same as Yuen's, but at third order he assumed a form of the solution which is devoid of secular terms. However, in carrying out the analytical details, we find that the assumed third-order solution does not satisfy the boundary conditions to the same order.

Rutland & Jameson (1971), following Yuen's analysis, performed experiments; they compared the jet profiles obtained experimentally with those predicted by Yuen's theory and found good agreement. Their experiments showed the existence of satellites with long to moderate wavelengths ($k = 0{\cdot}075$ to $k = 0{\cdot}683$).

In most liquid jet applications, it is necessary to electrically charge each drop and to achieve a varying amount of deflexion of the individually charged drops. Satellites, being usually small in size, have different deflexion sensitivity than the main drops. In general, it would be most desirable to eliminate the appearance of satellites entirely. The only way the presence of satellites may not be harmful is when a complete wavelength of liquid separates from the main stream as a single unit and then, even if satellites separate out, they later merge with the same parent drop (see figure 1).

This paper is the first in a series of three papers motivated by the need for eliminating or at least achieving a favourable configuration of satellites in a liquid jet. With present theories and experiments, the presence of the satellites has been found to be a dominant feature so long as a monochromic initial perturbation is applied to the jet. It was thought, therefore, that applying a non-monochromatic (a fundamental and one or more harmonics) disturbance may give the desired breakup shape of the liquid jet.

In all the previous theoretical work, the applied disturbance is assumed to be in the form of an amplitude distortion applied to the surface of the liquid column. However, most of the practical devices used to excite the disturbance in a jet do not create such an amplitude distortion of the outer surface of the liquid column. Rather, one induces a transverse velocity disturbance in the liquid column at the nozzle exit. Therefore, in the present analysis, we consider an initial-value problem in which the applied disturbance is in the form of an initial velocity field (of periodic nature, along the length of the column), and the initial surface profile is taken to be undisturbed.

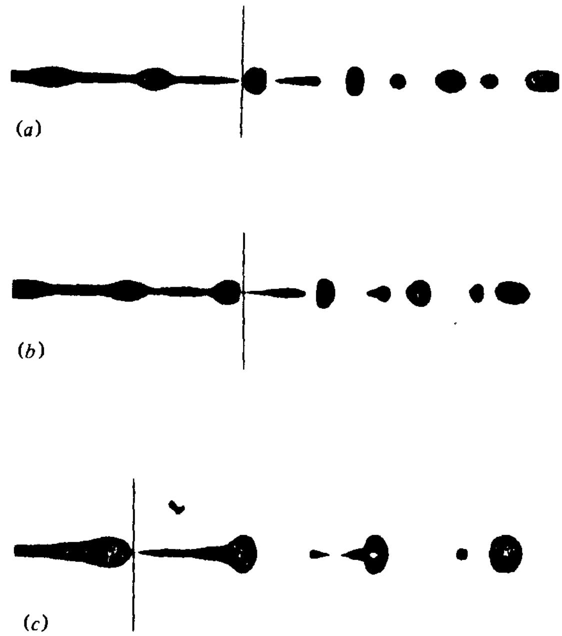

FIGURE 1. Typical ways of the satellite formation. (*a*) The satellite merging with the main drop following it; one wavelength of the fluid does not detach as one unit. (*b*) The satellite merging with the main drop leading it; one wavelength of the fluid does not detach as one unit. (*c*) One wavelength of the fluid detaches as one unit and the satellite merges with the parent drop, hence a desirable configuration.

2. Mathematical formulation and analysis

A long cylindrical column of an inviscid fluid is assumed to be initially at rest in a vacuum. The column has a uniform radius r_0 and the effect of any external field, including gravity, is neglected. The free surface of this column of fluid is subjected to a small normal velocity field which is axisymmetric, but periodic along the axis of the column (i.e. in the z direction). All variables are made dimensionless with respect to the characteristic length r_0 and the characteristic time $t_0 = (\rho r_0^3/\sigma)^{\frac{1}{2}}$, where ρ is the fluid density and σ the coefficient of surface tension. Using cylindrical polar coordinates, the dimensionless velocity potential $\phi(r, z, t)$ and the dimensionless surface distortion $\eta(z, t)$ are specified by the equations

$$\nabla^2\phi = 0, \quad 0 \leqslant r \leqslant 1+\eta, \tag{2}$$

$$\phi_r = \eta_t + \phi_z \eta_z \quad \text{on} \quad r = 1+\eta, \tag{3}$$

$$1-\phi_t - \tfrac{1}{2}(\phi_r^2+\phi_z^2) = (1+\eta_z^2)^{-\frac{1}{2}}\left\{\frac{1}{1+\eta} - \frac{\eta_{zz}}{1+\eta_z^2}\right\} \quad \text{on} \quad r = 1+\eta. \tag{4}$$

The initial conditions are

$$\eta(z, t=0) = 0, \tag{5}$$

$$\eta_t(z, t=0) = \epsilon\omega_0 \cos(k_0 z) + \delta_n \omega_0 \cos(nk_0 z + \theta), \tag{6}$$

where ω_0 is a characteristic scaling frequency, say the frequency determined from the linear dispersion relation

$$\omega_0^2 = \left|\frac{k_0(1-k_0^2)}{\alpha(k_0)}\right|, \quad \alpha(k_0) = \frac{I_0(k)}{I_1(k)}. \tag{7}$$

k_0 is the wavenumber and ϵ and δ_n are the amplitudes of the fundamental and the nth harmonic inputs. We assume that ϵ and δ_n are of comparable order, but both are small compared to the unit radius of the jet column. This permits a solution by perturbation methods and one for which the amplitude δ_n and phase θ, relative to the fundamental, can be adjusted to define 'optimal' experimental configurations. Incidentally, with no initial amplitude distortion, the conservation of mass condition (i.e. the mean location of the interface) is much easier to apply.

To construct the solution to the problem posed above, we express ϕ and η in parameter expansions of the form

$$\phi(r,z,t;\,\epsilon,\delta_n) = \sum_{m=1}^{\infty} \epsilon^m \phi_m(r,z,t;\,\delta), \tag{8}$$

$$\eta(r,t;\,\epsilon,\delta_n) = \sum_{m=1}^{\infty} \epsilon^m \eta_m(r,t;\,\delta), \tag{9}$$

where $\delta = \delta_n/\epsilon$ is the ratio of the initial inputs of the nth harmonic and the fundamental. Throughout the analysis we assume that δ is an order one parameter so that the fundamental and the harmonic are of (nearly) comparable amplitude initially. We require that these expansions are uniformly valid in the sense that

$$\frac{\eta_m}{\eta_{m-1}} < O(1) \quad \text{for} \quad \begin{Bmatrix} -\infty < z < \infty \\ 0 \leqslant t < T_1 \end{Bmatrix}, \tag{10}$$

where T_1 is infinite for stable oscillations of the interface and is related to the breakup time for amplified disturbances. As shown later, the time T_1 is comparable to the breakup time in a number of cases which we have considered. This is an important result which suggests that the following analysis should have some practical value in identifying the kind of jet forcing leading to 'acceptable' performance. To ensure condition (10), we also introduce the strained co-ordinates

$$\tau = t\left\{1 + \sum_{m=1}^{\infty} \epsilon^m \nu_m(\epsilon,\delta,k_0)\right\}, \tag{11a}$$

$$\zeta = z\left\{1 + \sum_{m=1}^{\infty} \epsilon^m k_m(\epsilon,\delta,k_0)\right\}, \tag{11b}$$

and then choose the straining functions ν_m and k_m to eliminate any secular terms arising in the expansion. In general, the straining functions depend on the independent variables also, but to the order considered here, they depend only on the parameters ϵ, δ and k_0 in the initial condition. Finally, to make the problem tractable, we expand the derivatives of ϕ appearing in the boundary conditions (3) and (4) in a Taylor series about $r = 1$ and then apply the conditions at $r = 1$ instead of at the unknown interface location $r = 1+\eta$.

2.1. *The linear problem*

The first-order problem is given by

$$(\partial^2_{rr}+r^{-1}\partial_r+\partial^2_{\zeta\zeta})\,\phi_1 = 0 \tag{12}$$

subject to the boundary conditions

$$\begin{aligned}\eta_{1,\tau} &= \phi_{1,r},\\ \phi_{1,\tau} &= \eta_1+\eta_{1,\zeta\zeta}\end{aligned} \tag{13}$$

at $r = 1$ and subject to the initial conditions

$$\begin{aligned}\eta_1(\zeta,0) &= 0,\\ \eta_{1,\tau}(\zeta,0) &= \omega_0[\cos k\zeta+\delta\cos(nk\zeta+\theta)].\end{aligned} \tag{14}$$

The wavenumber k in the strained co-ordinate system is related to k_0 through (11*b*):

$$k = k_0/(1+\epsilon k_1+\epsilon^2 k_2+\dots). \tag{15}$$

The solution to this order is similar to Rayleigh's result for the two independent inputs

$$\phi_1 = \omega_0\left[\frac{\alpha_1(k)}{k}\cos(k\zeta)\cosh(\omega_1\tau)+\delta\frac{\alpha_n(r)}{nk}\cos(nk\zeta+\theta)\cosh(\omega_n\tau)\right] \tag{16}$$

and

$$\eta_1 = \omega_0\left[\frac{\sinh(\omega_1\tau)}{\omega_1}\cos(k\zeta)+\frac{\sinh(\omega_n\tau)}{\omega_n}\cos(nk\zeta+\theta)\right], \tag{17}$$

where

$$\alpha_m(r) = \frac{I_0(mkr)}{I_1(mkr)},\quad \alpha_m = \frac{I_0(mk)}{I_1(mk)}, \tag{18}$$

and

$$\omega_m^2 = \frac{mk}{\alpha_m}(1-m^2k^2),\quad m = 1,2,n. \tag{19}$$

Depending on the value of k_0 in the initial condition, both solutions can be either amplified or neutral (oscillatory), or the fundamental can be amplified and the harmonic can be neutral. The nonlinear interactions can give other combinations and even extend the range of amplified wavenumbers. The results have been carried to third order where the fundamental and harmonic terms are regenerated by nonlinear interactions.

The first-order solution shows that, for $k < 1$, ω_1 is real and surface waves grow exponentially; for $k > 1$, ω_1 is pure imaginary and the surface waves oscillate. In a real fluid when $k > 1$, the waves will be damped because of the effect of the viscosity.

2.2. *The second-order problem*

The analysis is lengthy and tedious beyond the linear solution, so only skeletal results will be presented here. Detailed analytical expressions are given by Chaudhary (1977). The interface displacement at second order must be of the form

$$\begin{aligned}\eta_2 = {} & B_{21}(\tau)\cos(2k\zeta)+B_{22}(\tau)\cos[2(nk\zeta+\theta)]+B_{23}(\tau)\cos[(n-1)k\zeta+\theta]\\ & +B_{24}(\tau)\cos[(n+1)k\zeta+\theta]+B_{25}(\tau)\cos(k\zeta)+B_{26}(\tau)\cos(nk\zeta)+B_{20}(\tau).\end{aligned} \tag{20}$$

Solving for ϕ_2, which contains similar terms depending on ζ, and substituting into the second-order boundary conditions, one finds that both k_1 and ν_1 should be zero to eliminate any secular terms. Also, the initial conditions require that both B_{25} and B_{26} be identically zero. The form of the remaining coefficients $B_{2i}(i = 0, 1, \ldots, 4)$ are given in appendix A.

The second-order solution shows that there are four different wave components. Two of these represent twice the input frequencies (fundamental and nth harmonic) and the other two represent the interaction of the fundamental and the injected nth harmonic. Depending on the value of k and n, some or all of them can lie in the range where they show growing or oscillatory behaviour. There are several interesting features. For example if $n = 3$, the first interaction term is

$$\cos\left[(n-1)\,k\zeta+\theta\right] = \cos\left(2k\zeta+\theta\right),$$

which shows that in addition to $\cos 2k\zeta$, there is another second harmonic component, but at a different phase. Combining the two, $B_{21}(\tau)\cos 2k\zeta + B_{23}(\tau)\cos(2k\zeta+\theta)$, shows that the phase of the second harmonic can be time dependent. Similarly, for $n = 2$ there will be a fundamental component at a different phase than the input, showing that the phase of the combined fundamental can be time dependent. This time dependence of the phase of various wave components can provide control of the satellites. B_{20} is purely a time-dependent term in the second-order solution, and comes from volume conservation to the second order.

2.3. *The third-order problem*

At this order, the interface displacement must contain the following contributions:

$$\begin{aligned}\eta_3(z,t) = {} & B_{31}(\tau)\cos(k\zeta) + B_{32}(\tau)\cos(nk\zeta+\theta) + B_{33}(\tau)\cos(3k\zeta) \\ & + B_{34}(\tau)\cos\left[3(nk\zeta+\theta)\right] + B_{35}(\tau)\cos\left[(n-2)\,k\zeta+\theta\right] + B_{36}(\tau)\cos\left[(n+2)\,k\zeta+\theta\right] \\ & + B_{37}(\tau)\cos\left[(2n-1)\,k\zeta+2\theta\right] + B_{38}(\tau)\cos\left[(2n+1)\,k\zeta+2\theta\right]. \end{aligned} \tag{21}$$

After solving for ϕ_3 and substituting into the third-order boundary conditions, inhomogeneous, second-order, ordinary differential equations for the B_{3i} are obtained. The equations for B_{31} and B_{32} involve secular terms which can be eliminated by appropriately choosing the straining terms ν_2 and k_2. The algebraic details are very lengthy and not essential to the discussion which follows. Hence, we present the form of these coefficients in appendix B.

The third-order solution has eight different wave components. Four of them represent the interaction of individual inputs with their own harmonics, and the other four are the mutual interactions. Two important features are noted here.

(i) The $\cos(k\zeta)$ and $\cos(nk\zeta+\theta)$ terms show that there is a feedback to the original inputs at this order.

(ii) Interaction components $\cos[(n-2)\,k\zeta+\theta]$ and $\cos[(2n-1)\,k\zeta+2\theta]$ can be of the same wavelength as the fundamental or some of the harmonics produced by the fundamental. Therefore, the fundamental or some of the harmonics exhibit a time-dependent phase.

When examining different B_{2i}'s and B_{3i}'s in both the second- and third-order solutions, we see that, even if the wavenumber for a particular wave component is beyond the cut-off, it can show a growth rate due to the interaction with lower-order components. This means that, if the fundamental input has a growing solution, all the harmonic components will have some growth associated with them, irrespective of whether their own wavenumber is below the cut-off. In particular, it is interesting to note that, if n is high enough such that nk is beyond the cut-off, even then the included nth harmonic has some growth rate associated with it provided that the fundamental has a growing solution. This is due to the feedback of nth harmonic component from the third-order solution. Furthermore, all the harmonics ($2n$, $3n$, etc.) of the included nth harmonic will have some growth associated with them due to interaction with the fundamental. If the wavenumber of a particular harmonic is beyond the cut-off, its growth is sustained only by direct feeding of the energy by the lower harmonics or the fundamental.

3. Cut-off wavenumber

The cut-off wavenumber and the solution near it are examined for the fundamental input only (i.e. $\delta = 0$). This also gives the range of wavenumbers in which the solution is expected to be valid. Wavenumbers in the physical and the strained coordinate systems are related by

$$k = k_0/(1+\epsilon^2 k_2). \tag{22}$$

Selecting the first choice of k_2 [equation (B 6)] and noting that ω_0 is just a positive quantity given by (7), we obtain

$$(1-k^2)(k-k_0) = \frac{9}{16}\frac{\alpha_1(k)}{\alpha_1(k_0)}\epsilon^2 \left|k_0(1-k_0^2)\right|. \tag{23}$$

Examining this, the following relations are noted:

(i) when $k_0 = 1$, $k = 1$; (24)

(ii) when $k_0 < 1$, $k_0 < k < 1$; (25)

(iii) when $k_0 > 1$, $1 < k < k_0$. (26)

The other two choices of k_2 give exactly the same relations between k and k_0. Since the third-order solution obtained grows exponentially for $k < 1$ and is oscillatory for $k > 1$, the cut-off wavenumber is $k = 1 = k_0$.

To examine the behaviour in the cut-off region, we must choose ω_0 as a fixed constant instead of using the linear dispersion result (7) so that the disturbance amplitude does not vanish at $k = 1$. Now $(\omega_0\epsilon)$ is just the magnitude of the initial velocity disturbance independent of k_0 and (23) becomes

$$(1-k^2)(\,(k-k_0) = \frac{9}{16}\frac{\alpha_1(k)}{\alpha_1(k_0)}(\omega_0\epsilon)^2. \tag{27}$$

The right-hand side is always greater than zero. Examining the equation we find the following:

(i) when $k_0 = 1$, no real solution for k exists;

when k_0 is such that the real solution for k exists, and

(ii) when $k_0 < 1$, $k_0 < k < 1$;

(iii) when $k_0 > 1$, $1 < k < k_0$.

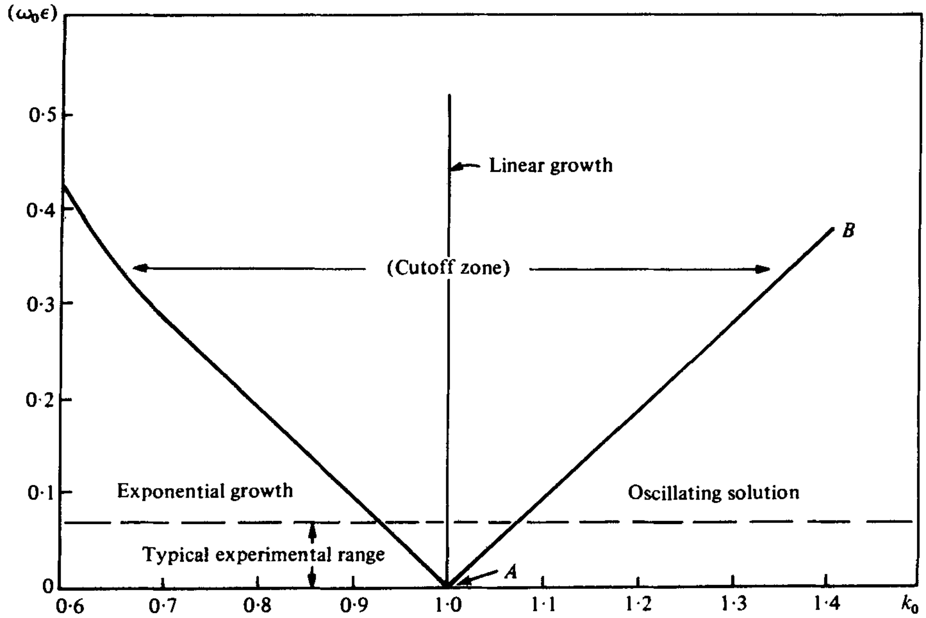

FIGURE 2. Three zones of solution for finite initial input.

The last two relations are the same as before except that, for finite initial input, real k does not exist in the neighbourhood of $k_0 = 1$, implying that the solution developed can not be applied in that neighbourhood. To find the range of k_0 where the above solution does not apply for a given $(\omega_0\epsilon)$, the extreme values of k_0 are found from (27) such that k is real. Treating k as a parametric variable, the extremum of k_0 and the corresponding value of $(\omega_0\epsilon)$ are given in parametric form:

$$k_0 = \frac{k\{2k\alpha_1(k) + (1-k^2)[1-\alpha_1^2(k)]\}}{[\alpha_1(k)](1/k+k) + (1-k^2)[1-\alpha_1^2(k)]}, \tag{28}$$

$$(\omega_0\epsilon)^2 = \frac{16}{9}[\alpha_1(k_0)/\alpha_1(k)](1-k^2)(k-k_0). \tag{29}$$

When k is less than one, (28) gives the maximum value of k_0 and, when k is greater than one, it gives the minimum value of k_0. Equation (29) gives the corresponding value of $\omega_0\epsilon$.

Figure 2 shows the zones of k_0 for finite initial input $(\omega_0\epsilon)$, where the solution developed will apply together with the cut-off zone. In the cut-off zone, special care must be taken in interpreting the solution recorded in § 2.

3.1. *The solution near $k_0 = 1$*

To first order and for finite initial input, the solution is written

$$\eta(z,t;\,\omega_0\epsilon) = (\omega_0\epsilon)\frac{\sinh(\omega_1 t)}{\omega_1}\cos(k_0 z), \tag{30}$$

where ω_0 is just a constant and ω_1 is given by

$$\omega_1 = k_0(1-k_0^2)/\alpha_1(k_0).$$

Note that when $k_0 \to 1$, $\omega_1 \to 0$ and

$$\eta(z, t;\, \omega_0 \epsilon)\big|_{k_0=1} = (\omega_0 \epsilon)\, t \cos(k_0 z). \tag{31}$$

This shows that near $k_0 = 1$ there is a linear growth instead of exponential growth. The linear growth at $k_0 = 1$ is one of the essentially different features of the initial velocity problem in comparison with the initial amplitude problem. In the case of an initial amplitude problem as shown by Yuen (1968), the first-order solution is stationary at $k_0 = 1$. The result for an initial velocity input can be explained by considering the first-order solution only. In an initial amplitude problem, the surface is deformed and the liquid column is at rest everywhere initially. Since there is no velocity and if the initial deformation has such a curvature that the inside pressure is in balance with the surface tension forces, there need not be any subsequent motion of the fluid, resulting in a stationary solution. However, when an initial velocity is applied to a liquid column of uniform radius, the interface begins to move with a displacement which is linear in time, even though the surface tension forces are in balance with the pressure initially.

We can expect that the linearly growing solution at $k_0 = 1$ will apply in a small band of wavenumbers in the cut-off zone around $k_0 = 1$. It is interesting to note from figure 2 that, if k_0 is slightly less than one for very small initial input, the growth rate is exponential, but as the magnitude of the initial disturbance is increased, for the same wavenumber, the growth rate will tend to become linear in time. Similarly, for k_0 slightly greater than one and for small initial input, the surface just oscillates and the column does not break into discrete drops. As the initial input increases, the disturbance grows linearly in time and the column will break into discrete drops.

For a constant initial disturbance $(\omega_0 \epsilon)$, if the wavenumber is increased from small values to $k_0 > 1$, we observe from figure 2 that the solution form changes from one exhibiting exponential growth to one with linear growth and then to one with neutral oscillatory motion. This shows that there is no sharp cut-off wavenumber for the initial velocity input. This is in contrast to the sharp cut-off wavenumber given by Yuen (1968) and Nayfeh (1970) for the initial amplitude input. However, we choose to refer to the line AB in figure 2 as the cut-off wavenumber. This line is given by equations (28) and (29) and can be approximated to

$$k_{\text{cut-off}} = 1 + 1{\cdot}0665(\omega_0 \epsilon). \tag{32}$$

4. Theory of satellite control with harmonic input

In this section we show theroretically how the input harmonic in the initial condition may effect the formation of satellites. Most frequently, only one satellite is observed whose formation is primarily due to the second-harmonic component arising through the nonlinearity in the time evolution of the motion. The following discussion is addressed to this case.

We observe from the solution that one second-harmonic component always comes from the fundamental input in second order. Another second-harmonic component can enter in either of two ways:

(i) the input harmonic itself may be a second harmonic; or,

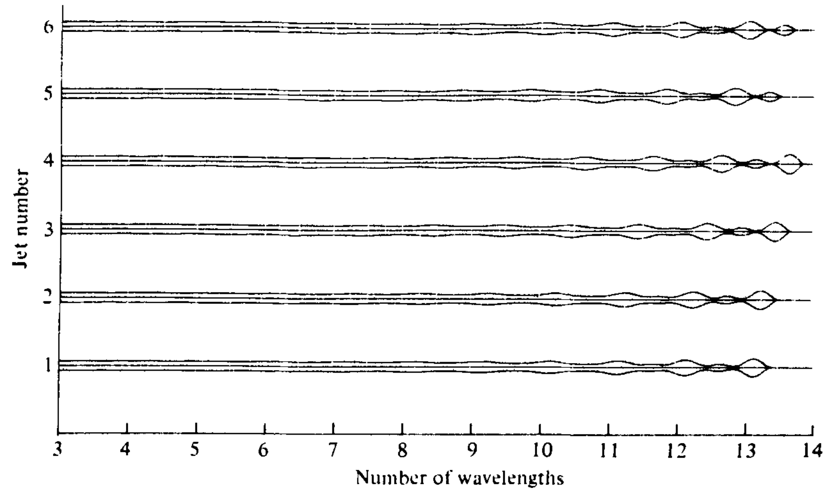

FIGURE 3. Theoretical jet profile with fundamental input only, observed at different instances. $k_0 = 0{\cdot}4312$, $\epsilon = 0{\cdot}01$, $\delta = 0$, $n = 3$, $\theta = 0$, non-dimensional frequency $= 0{\cdot}694$.

Jet number	Time (break)
1	19·2939
2	19·3948
3	19·6398
4	19·8991
5	19·4524
6	19·7262

(ii) the interaction of the input harmonic with the fundamental may give a second-harmonic component. For example, with the third-harmonic input ($n = 3$) the second-order solution [equation (20)] gives an interaction term $B_{23}\cos(2k\zeta+\theta)$.

Consider first the case with third-harmonic input whereby two second-harmonic terms appear in the second-order solution [equation (20)]. They can be combined to give

$$B_{21}\cos 2k\zeta + B_{23}\cos(2k\zeta+\theta) = \bar{B}_2\cos(2k\zeta+\bar{\theta}), \tag{33}$$

where

$$\bar{B}_2(\tau,\theta) = [(B_{21}+B_{23}\cos\theta)^2+(B_{23}\sin\theta)^2]^{\frac{1}{2}}, \tag{34}$$

$$\bar{\theta}(\tau,\theta) = \tan^{-1}\frac{B_{23}\sin\theta}{B_{21}+B_{23}\cos\theta}. \tag{35}$$

$\bar{B}_2(\tau,\theta)$ and $\bar{\theta}(\tau,\theta)$ are the time-dependent magnitude and phase (in relation to the fundamental) of the composite second harmonic. It should be noted that, if the fundamental has a growing solution ($k_0 < 1$), both B_{21} and B_{23} will be growing terms and so will the magnitude $\bar{B}_2$. Except then $\sin\theta = 0$ and/or $B_{23} = 0$, the composite second-harmonic will always have a time-dependent phase in relation to the fundamental. This shows that θ can be selected appropriately to make $\bar{\theta}$ increasing or decreasing with time. In the case when there is only one satellite and assuming it is primarily governed by the second-harmonic component, the above analysis shows that the formation of the satellite can be controlled by changing the harmonic input (δ_n and θ).

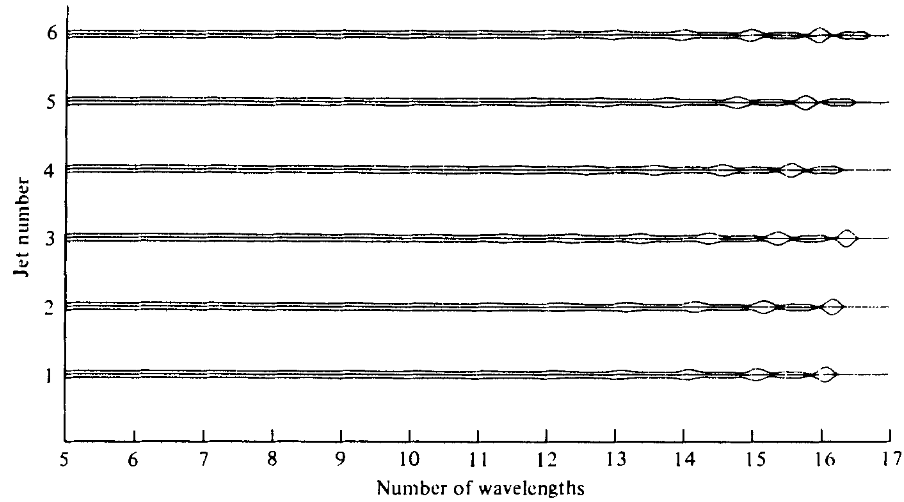

FIGURE 4. Theoretical jet profile with fundamental input only, observed at different instances. $k_0 = 0{\cdot}31$, $\epsilon = 0{\cdot}01$, $\delta = 0$, $n = 3$, $\theta = 0$, non-dimensional frequency $= 0{\cdot}694$.

Jet number	Time (break)
1	23·4150
2	23·5447
3	23·8040
4	23·5158
5	23·7752
6	24·0490

A rough comparison can be made on the effectiveness of either second- or third-harmonic input in controlling the satellites. When the initial input includes the second harmonic, the controlling term will come from the first and the third-order solutions [i.e. B_{32} in (21)]. If k_0 is such that $2k_0$ is in the non-growing range or has a very small growth rate, the controlling term will be small in magnitude even though one part of it is coming from the first-order solution. On the other hand, when the initial input includes the third harmonic, the controlling term comes from the second-order solution and has a growing component in it (because it is an interaction term with the fundamental). For large values of time (τ), the magnitude of this second-order interaction term may become larger than the first-order (non-growing) second-harmonic term showing that, in some range of k_0, third-harmonic input may be more effective than second-harmonic input.

5. Numerical computation for a progressive jet

In case of a jet issuing from a nozzle, the solution described above does not apply directly. The initial disturbance in such a progressive jet can be applied either upstream from the nozzle or a short distance after the nozzle. In an actual jet, the velocity profile is not exactly uniform. However, substantial growth of the surface waves takes place on the portion of the jet where the velocity profile is nearly uniform. A uniform mean velocity is assumed across the section of the jet at the point of application of the

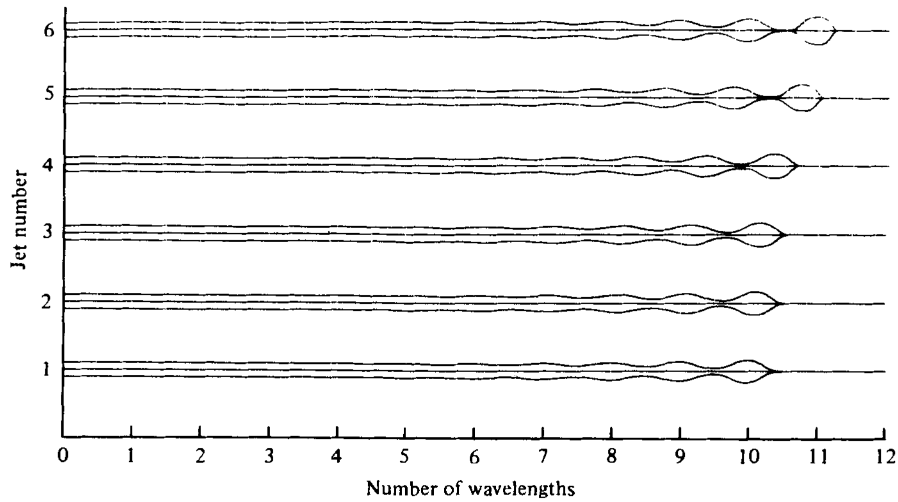

FIGURE 5. Theoretical jet profile with fundamental input only observed at different instances. $k_0 = 0{\cdot}65$, $\epsilon = 0{\cdot}01$, $\delta = 0$, $n = 3$, $\theta = 0$, non-dimensional frequency $= 0{\cdot}694$.

Jet number	Time (break)
1	15·1441
2	15·1441
3	15·1729
4	15·3746
5	15·8646
6	16·1239

initial disturbance. Distance down stream from this point can be related to time through the mean jet velocity. To apply the result of an infinite liquid column to a progressive jet, the following co-ordinate transformation is used. If (r, z, t) is the co-ordinate system for an infinite column, and (r, x, T) is the co-ordinate system for a progressive jet, the transformation relating the two systems is $z = x - V_0 T$ and $t = x/V_0$, where V_0 is the (mean) velocity of the undisturbed jet. By selecting T, the instantaneous profiles of the jet surface are calculated from the point of application of the disturbances ($x = 0$) to the point where the jet breaks into drops ($\eta = -1$).

Figures 3, 4 and 5 show computed jet profiles for a fundamental input alone for moderate, low and high wavenumbers. The profiles are plotted for six different instances. For moderate wavenumbers ($k_0 = 0{\cdot}431$, figure 3) we find sequentially a large and a small (satellite) drop detached from the jet showing one satellite, while for short wavenumbers ($k_0 = 0{\cdot}31$, figure 4) the satellite is a long filament with a neck in between showing the possibility that it may subdivide in two satellites further downstream. For large wavenumbers ($k_0 = 0{\cdot}65$, figure 5) only a single drop separated out with a very thin ligament attached to it. Since the solution developed here cannot be applied beyond the breakup point, it cannot predict whether the thin ligament will coalesce with the drop or separate out as a small satellite.

To determine the limitations of the analysis, we examine how far downstream these results are valid in the sense that relation (10) is satisified. Figures 6, 7 and 8 show magnitudes of the fundamental, second and third harmonics for the same initial

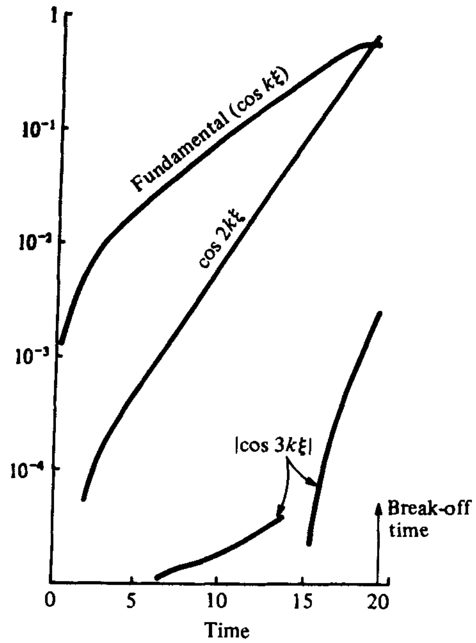

FIGURE 6

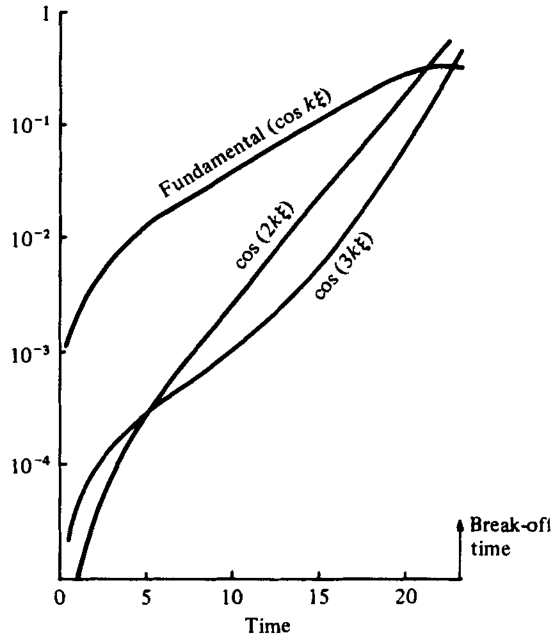

FIGURE 7

FIGURE 6. Magnitudes of the fundamental, second and third harmonics for the fundamental input only, for $k_0 = 0{\cdot}4321$ and $\epsilon = 0{\cdot}01$.

FIGURE 7. Magnitudes of the fundamental, second and third harmonics for the fundamental input only for $k_0 = 0{\cdot}31$ and $\epsilon = 0{\cdot}01$.

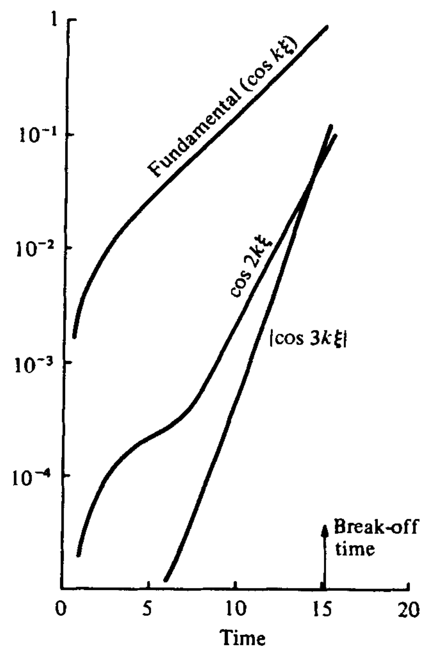

FIGURE 8. Magnitudes of the fundamental, second and third harmonics for the fundamental input only for $k_0 = 0{\cdot}65$ and $\epsilon = 0{\cdot}01$.

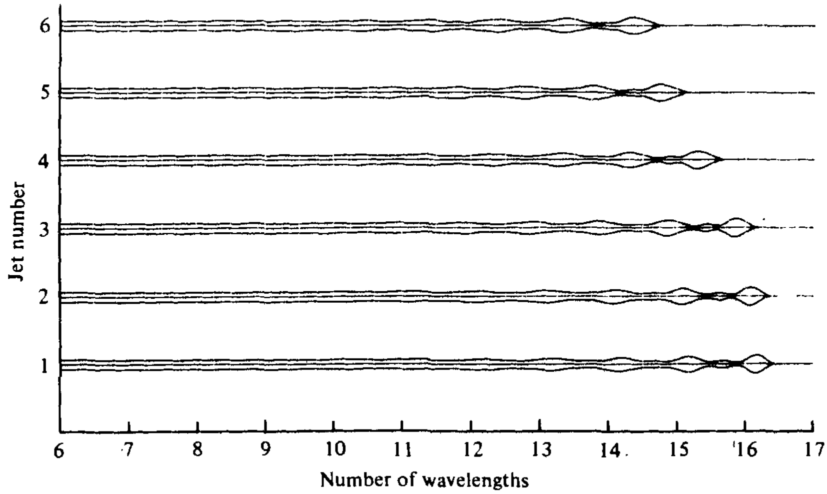

FIGURE 9. Theoretical jet profile with the third harmonic input. $k_0 = 0{\cdot}4312$, $\epsilon = 0{\cdot}00306$, $n = 3$, $\theta = 90$, non dimensional frequency $= 0{\cdot}694$.

Jet number	δ	Time (break)
1	0·0	23·6743
2	0·0500	23·5879
3	0·1000	23·3285
4	0·1400	22·5792
5	0·1800	21·8300
6	0·2200	21·2536

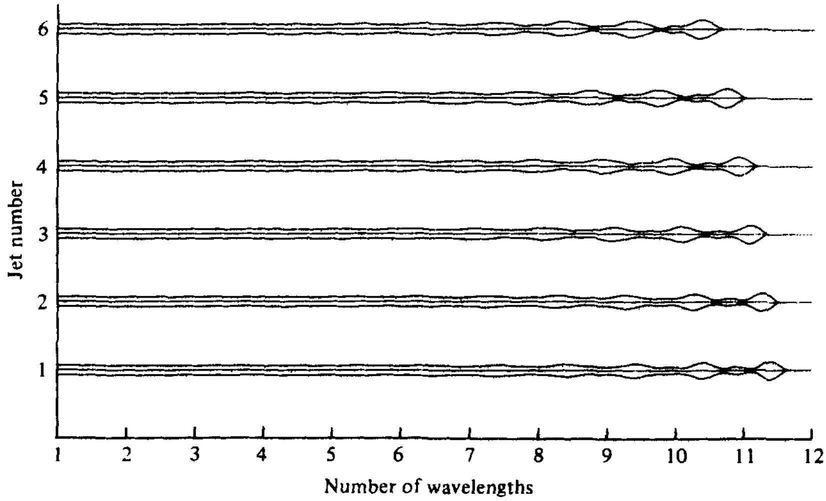

FIGURE 10. Theoretical jet profile with third harmonic input. $k_0 = 0{\cdot}4312$, $\epsilon = 0{\cdot}01946$, $n = 3$, $\theta = 90$, non-dimensional frequency $= 0{\cdot}694$.

Jet number	δ	Time (break)
1	0·0	16·8156
2	0·0500	16·6138
3	0·1000	16·3833
4	0·1400	16·1527
5	0·1800	15·9222
6	0·2200	15·4611

input as in the jets of figures 3, 4 and 5 respectively. We observe (figure 7) that for small wavenumbers case ($k_0 = 0{\cdot}31$) the magnitudes of the harmonics do not exceed that of the fundamental until 91 % of the break-off time. For $k_0 = 0{\cdot}431$ (figure 6) the results are valid up to 97 % of the break-off time and for larger wavenumbers ($k_0 = 0{\cdot}65$, figure 8) the harmonics never exceed the fundamental. Several more computer runs were made and it was found that for a wavenumber about 0·51 or large the magnitude of the harmonics never exceeded that of the fundamental.

Figures 9 and 10 show computed jet profiles for the moderate wavenumber case with both fundamental and third-harmonic inputs. Six profiles are shown in these figures for different initial amplitudes of the harmonic inputs relative to the fundamental. We observe that with sufficient harmonic input we can change the shape near the breakup point to the extent that satellites can be eliminated.

Appendix A

The form of the solution for the functions $B_{2i}(\tau)$, $(i = 0, 1, \ldots, 4)$ appearing in (20) are as follows:

$$B_{20} = b_{200} + b_{201} \cosh(2\omega_1\tau) + b_{202}\cosh(2\omega_n\tau),$$
$$B_{21} = b_{210} + b_{211} \cosh(2\omega_1\tau) + b_{212}\cosh(\omega_2\tau),$$
$$B_{22} = b_{220} + b_{221} \cosh(2\omega_n\tau) + b_{222}\cosh(\omega_{2n}\tau),$$
$$B_{23} = b_{230}\cosh(\omega_{n-1}\tau) + b_{231}\frac{\sinh(\omega_1\tau)}{\omega_1}\frac{\sinh(\omega_n\tau)}{\omega_n} + b_{232}\cosh(\omega_1\tau)\cosh(\omega_n\tau),$$
$$B_{24} = b_{240}\cosh(\omega_{n+1}\tau) + b_{241}\frac{\sinh(\omega_1\tau)}{\omega_1}\frac{\sinh(\omega_n\tau)}{\omega_n} + b_{242}\cosh(\omega_1\tau)\cosh(\omega_n\tau).$$

The coefficients b_{2ij} have the following definitions:

$$b_{201} = -\tfrac{1}{8}(\omega_0^2/\omega_1^2),$$
$$b_{202} = -\tfrac{1}{8}(\delta^2\omega_0^2/\omega_n^2),$$
$$b_{200} = -(b_{201}+b_{202}),$$
$$b_{210} = -\frac{k}{4\alpha_2\omega_2^2}\frac{\omega_0^2}{\omega_1^2}[\omega_1^2(1+\alpha_1^2)+2+k^2],$$
$$b_{211} = \frac{1}{2}\frac{\omega_0^2/\omega_1^2}{\omega_2^2-4\omega_1^2}\left\{\frac{k}{2\alpha_2}[\omega_1^2(3-\alpha_1^2)+2+k^2]+\omega_1^2(1-2k\alpha_1)\right\},$$
$$b_{212} = -(b_{210}+b_{211}),$$
$$b_{220} = -\frac{nk\delta^2}{4\alpha_{2n}\omega_{2n}^2}\frac{\omega_0^2}{\omega_n^2}[\omega_n^2(1+\alpha_n^2)+2+(nk)^2],$$
$$b_{221} = \frac{1}{2}\frac{\omega_0^2\delta^2/\omega_n^2}{\omega_{2n}^2-4\omega_n^2}\left\{\frac{nk}{2\alpha_{2n}}[\omega_n^2(3-\alpha_n^2)+2+(nk)^2]+\omega_n^2(1-2nk\alpha_n)\right\},$$
$$b_{222} = -(b_{220}+b_{221});$$
$$b_{231} = (D_{n-1})^{-1}\{(\omega_{n-1}^2-\omega_n^2-\omega_1^2)A_{n-1}^{(1)}+2\omega_1^2\omega_n^2A_{n-1}^{(2)}\},$$
$$b_{230} = -b_{232} = -(D_{n-1})^{-1}\{(\omega_{n-1}^2-\omega_n^2-\omega_1^2)A_{n-1}^{(2)}+2A_{n-1}^{(1)}\},$$
$$b_{241} = (D_{n+1})^{-1}\{\omega_{n+1}^2-\omega_n^2-\omega_1^2)A_{n+1}^{(1)}+2\omega_1^2\omega_n^2A_{n+1}^{(2)}\},$$
$$b_{240} = -b_{242} = -(D_{n-1})^{-1}\{\omega_{n+1}^2-\omega_n^2-\omega_1^2)A_{n+1}^{(2)}+2A_{n+1}^{(1)}\},$$

where

$$D_{n\pm1} = (\omega_{n\pm1}^2 - \omega_n^2 - \omega_1^2)^2 - 4\omega_1^2\omega_n^2,$$

$$A_{n\pm1}^{(1)} = \frac{(n\pm1)\,k}{\alpha_{n\pm1}}\frac{\omega_0^2\delta}{2}(\omega_1^2+\omega_n^2+1\pm nk^2) + \omega_n^2\frac{\omega_0^2\delta}{2}[1-(n\pm1)\,k\alpha_n)]$$
$$+\,\omega_1^2\frac{\omega_0^2\delta}{2}[1\mp(n\pm1)\,k\alpha_1],$$

$$A_{n\pm1}^{(2)} = \frac{(n\pm1)\,k}{\alpha_{n\pm1}}\frac{\omega_0^2\delta}{2}(1\mp\alpha_1\alpha_n) + \frac{\omega_0^2\delta}{2}[1-(n\pm1)\,k\alpha_n] + \frac{\omega_0^2}{2}[1\mp(n\pm1)\,k\alpha_1].$$

Appendix B

Equations for B_{31} and B_{32} which involve secular terms in the third-order solution are

$$\ddot{B}_{31} - \omega_1^2 B_{31} = k_2\omega_0\omega_1 k(\alpha_1 - 1/\alpha_1)\sinh(\omega_1\tau) - \dot{P}_{31}(\tau) - kQ_{31}(\tau)/\alpha_1, \tag{B 1}$$

$$\ddot{B}_{32} - \omega_n^2 B_{32} = k_2\omega_0\delta\omega_n nk(\alpha_n - 1/\alpha_n)\sinh(\omega_n\tau) - \dot{P}_{32}(\tau) - nkQ_{32}(\tau)/\alpha_n, \tag{B 2}$$

where P_{3j} and Q_{3j} involve hyperbolic functions of $\omega_i\tau$ $(i = 1, 2, n, 2n, n-1, n+1)$ and are given by Chaudhary (1977). The particular solution for the sinh $(\omega_1\tau)$ term in the equation for B_{31} will be of the form $\tau\sinh(\omega_1\tau)$. Similarly, the solution of B_{32} will have a term $\tau\sinh(\omega_n\tau)$. These secular terms are suppressed by choosing the straining terms k_2 and ν_2 such that their coefficients become zero. Solving for k_2 and ν_2 yields the expressions

$$k_2 = (\omega_1^2G_2 - \omega_n^2 g_2)/(\omega_1^2 G_1 - \omega_n^2 g_1), \tag{B 3}$$

$$\nu_2 = \tfrac{1}{2}(G_1g_2 - g_1G_2)/(\omega_1^2G_1 - \omega_n^2g_1), \tag{B 4}$$

where

$$g_1 = 2k^3/\alpha_1 - k\omega_1^2(\alpha_1 - 1/\alpha_1),$$

$$g_2 = -(kq'_{311}/\alpha_1 + \omega_1^2 p'_{311})/\omega_0,$$

$$G_1 = 2n^3k^3/\alpha_n - nk\omega_n^2(\alpha_n - 1/\alpha_n),$$

$$G_2 = -(nkq'_{321}/\alpha_n + \omega_n^2 p'_{321})/(\omega_0\delta),$$

$$\begin{aligned}
q'_{311} = {} & \tfrac{1}{4}\omega_0[2\omega_1^2 b_{211} + \tfrac{1}{4}\omega_0^2(1-2k\alpha_1)](\alpha_1\alpha_2 - 1) + \tfrac{1}{4}\omega_0(\omega_1^2 + 2 - 2k^2)(2b_{210} - b_{211}) \\
& + \tfrac{1}{2}\omega_0(\omega_1^2+2)(2b_{200} - b_{201}) \\
& - \frac{\delta\omega_0}{4\omega_n^2}\{[\omega_1^2 + 2\omega_n^2 + 2 - n(n-1)\,k^2]\,b_{231} + 2\omega_1^2\omega_n^2(b_{232} + b_{242}) \\
& + [\omega_1^2 + 2\omega_n^2 + 2 - n(n+1)\,k^2]\,b_{241} + \delta\omega_0^2\omega_n^2(1 - nk\alpha_n) + \delta\omega_0^2\omega_1^2(1-k\alpha_1)\} \\
& - \frac{\omega_0}{32}\frac{\omega_0^2}{\omega_1^2}[\omega_1^2(k\alpha_1 - 3) + 3(k^2 - 6 - 3k^4)] \\
& + \frac{\omega_0}{8}\frac{\omega_0^2\delta^2}{\omega_n^2}[2nk\alpha_n\omega_n^2 + \omega_1^2(1-k\alpha_1) + n^2k^2(3k^2-1) + 6] \\
& + \tfrac{1}{4}\omega_0\delta\{b_{231} + \omega_1^2 b_{232} + \tfrac{1}{2}\omega_0^2\delta[1-(n-1)\,k\alpha_n]\}(1 + \alpha_n\alpha_{n-1}) \\
& + \tfrac{1}{4}\omega_0\delta\{b_{241} + \omega_1^2 b_{242} + \tfrac{1}{2}\omega_0^2\delta[1-(n+1)\,k\alpha_n]\}(1 + \alpha_n\alpha_{n+1}),
\end{aligned}$$

$$
\begin{aligned}
p'_{311} =\ & \tfrac{1}{4}\omega_0[2b_{211}(k\alpha_2-1)+(b_{211}+2b_{210})(1+k\alpha_1)]+\tfrac{1}{2}\omega_0(b_{201}+2b_{200})(1-k\alpha_1)\\
&+\frac{\omega_0^3}{32\omega_1^2}[k^2(1-4\alpha_1\alpha_2)+k(\alpha_1+2\alpha_2)+4]-\tfrac{1}{4}\omega_0\delta k(\alpha_n+\alpha_{n-1})b_{232}\\
&+\tfrac{1}{4}\omega_0\delta k(\alpha_n+\alpha_{n+1})b_{242}-\frac{\omega_0\delta}{4\omega_n^2}[b_{231}(1+k\alpha_{n-1})+b_{241}(1-k\alpha_{n+1})]\\
&-\frac{\omega_0^3\delta^2}{8\omega_n^2}\{[1+(n-1)k\alpha_1](1+k\alpha_{n-1})+[1-(n+1)k\alpha_1](1-k\alpha_{n+1})\}\\
&+\frac{\omega_0^3\delta^2}{8\omega_n^2}(k^2+2-k\alpha_1),
\end{aligned}
$$

$$
\begin{aligned}
q'_{321} =\ & \tfrac{1}{4}\omega_0[2\omega_n^2b_{221}+\tfrac{1}{4}\omega_0^2\delta^2(1-2nk\alpha_n)](\alpha_n\alpha_{2n}-1)\\
&+\tfrac{1}{4}\omega_0\delta(\omega_1^2+2-2n^2k^2)(2b_{220}-b_{221})+\tfrac{1}{2}\omega_0\delta(\omega_n^2+2)(2b_{200}-b_{202})\\
&-\frac{\omega_0}{4\omega_1^2}\{[\omega_n^2+2\omega_1^2+2+(n-1)k^2]b_{231}+2\omega_1^2\omega_n^2(b_{232}+b_{242})\\
&+[\omega_n^2+2\omega_1^2-(1+n)k^2]b_{241}+\delta\omega_0^2\omega_n^2(1-nk\alpha_n)+\delta\omega_0^2\omega_1^2(1-k\alpha_1)\}\\
&-\frac{\delta^3\omega_0^3}{32\omega_n^2}[\omega_n^2(nk\alpha_n-3)+3(n^2k^2-6-3n^4k^4)]\\
&+\frac{\delta\omega_0}{8}\frac{\omega_0^2}{\omega_1^2}(2k\alpha_1\omega_1^2+\omega_n^2(1-nk\alpha_n)+k^2(3n^2k^2-1)+6]\\
&+\tfrac{1}{4}\omega_0\{b_{231}+\omega_n^2b_{232}+\tfrac{1}{2}\omega_0^2\delta[1+(n-1)k\alpha_1]\}(1-\alpha_n\alpha_{n-1})\\
&+\tfrac{1}{4}\omega_0\{b_{241}+\omega_n^2b_{242}+\tfrac{1}{2}\omega_0^2\delta[1-(n+1)k\alpha_1]\}(1+\alpha_n\alpha_{n+1}),
\end{aligned}
$$

$$
\begin{aligned}
p'_{321} =\ & \tfrac{1}{4}\omega_0[-nk(\alpha_1-\alpha_{n-1})b_{232}+nk(\alpha_1+\alpha_{n+1})b_{242}]\\
&-\frac{\omega_0}{4\omega_1^2}[(1-nk\alpha_{n-1})b_{231}+(1-nk\alpha_{n+1})b_{241}]\\
&-\frac{\omega_0^3\delta}{8\omega_1^2}\{[1-(n-1)k\alpha_n](1-nk\alpha_{n-1})+[1-(n+1)k\alpha_n](1-nk\alpha_{n+1})\}\\
&+\tfrac{1}{4}\omega_0\delta[2b_{221}(nk\alpha_{2n}-1)+(b_{221}-2b_{220})(1+nk\alpha_n)]\\
&+\tfrac{1}{2}\omega_0\delta(b_{202}+2b_{200})(1-nk\alpha_n)\\
&+\frac{\omega_0^3\delta^3}{32\omega_n^2}[n^2k^2(1-4\alpha_n\alpha_{2n})+nk(\alpha_n+2\alpha_{2n})+4]+\frac{\omega_0^3\delta}{8\omega_1^2}(n^2k^2+2-nk\alpha_n).
\end{aligned}
$$

Examination of these terms reveals that both ν_2 and k_2 are finite for all values of k, including the linear cut-off wavenumber $k = 1$, except when the harmonic input (i.e. δ) vanishes, whereupon G_2 is singular. However, when there is no harmonic input, all the $nk\zeta$ terms and its interactions are absent leaving only a single equation [the equation for $B_{31}(\tau)$] to determine both ν_2 and k_2. Following an argument similar to that used by Yuen (1968), ν_2 is made unique by requiring it to be finite for all values of k. However, there are three possible choices for k_2 which accomplish this and the appropriate one may require consideration of higher-order terms in the expansion.

When there is no harmonic input, the only equation giving rise to secular growth is (B 1). Choosing the coefficient ν_2 to eliminate this secularity we obtain

$$\begin{aligned}\nu_2 = {} & \tfrac{1}{2}k_2 k(\alpha_1 - 1/\alpha_1) + \tfrac{1}{8}(1/\omega_1^2)\,[2\omega_1^2 b_{211} + \tfrac{1}{4}\omega_0^2(1-2k\alpha_1)]\,(1-2k\alpha_2 + k/\alpha_1) \\ & - \tfrac{1}{8}(b_{211} + 2b_{210})\,(1+k\alpha_1) + \tfrac{1}{8}(b_{211} - 2b_{210})\,(2 + k/\alpha_1) \\ & - \tfrac{1}{64}(\omega_0^2/\omega_1^2)\,(8 - 5k\alpha_1 + 9k/\alpha_1) \\ & - \tfrac{1}{64}(\omega_0^2/\omega_1^4)\,k/\alpha_1[(30 + 9k^4 - 3k^2) + 64k_2 k^2\omega_1^2/\omega_0^2]. \end{aligned} \tag{B 5}$$

At $k = 1$ both ω_1 and ω_0 are zero. All the factors in ν_2 remain finite except ω_0^2/ω_1^4, which is singular. To make ν_2 finite at $k = 1$ the value of k_2 is adjusted so that the term in the last square bracket has

(i) a factor of ω_1^2 in the numerator, or

(ii) its value is zero.

This allows three possible choices for k_2 as shown below.

(i) $$k_2 = -\tfrac{9}{16}(\omega_0^2/\omega_1^2); \tag{B 6†}$$

this reduces the last term in ν_2 to $-\tfrac{1}{64}(30 - 9k^2)\,(\omega_0^2/\omega_1^2)$.

(ii) $$k_2 = -\tfrac{1}{64}(27 + 9k^2)\,(\omega_0^2/\omega_1^2); \tag{B 7}$$

this reduces the last term in ν_2 to $-(15/32)\,(\omega_0^2/\omega_1^2)$.

(iii) $$k_2 = -\frac{3}{64k^2}(10 + 3k^4 - k^2)\,(\omega_0^2/\omega_1^2); \tag{B 8}$$

this reduces the last term in ν_2 to zero.

The remaining terms in the third-order solution given in (21) are very lengthy and the interested reader is referred to Chaudhary (1977) for specific forms for all the coefficient functions $B_{3i}(\tau)$, $i = 1$ to 8. They have been used in the numerical results presented in § 5, but are not essential to an understanding of the principal results concerning satellite control.

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† This choice of k_2 corresponds to Yuen's selection of his k_{c3}. The negative sign and the factor of (ω_0^2/ω_1^2) are due to the fact that problem here is an initial velocity problem instead of the initial amplitude problem solved by Yuen.